
6.3 Two-Point Correlation

The Kolmogorov hypotheses, and deductions drawn from them, have no direct connection to the Navier-Stokes equations (although, as in the previous section, the continuity equation is usually invoked). And although in the description of the energy cascade, the energy transfer to successively smaller scales has been identified as a phenomenon of prime importance, the precise mechanism by which this transfer takes place has not been identified or quantified. It is natural, therefore, to try to extract from the Navier-Stokes equations useful information about the energy cascade. The earliest attempts (outlined in this section) are those of Taylor (1935a) and of von Kármán and Howarth (1938), which are based on the two-point correlation. The next two sections gives the view from wavenumber space in terms of the energy spectrum—the Fourier transform of the two-point correlation.

Autocorrelation Functions. Consider homogeneous isotropic turbulence, with zero mean velocity, r.m.s. velocity $u'(t)$ and dissipation rate $\varepsilon(t)$. Because of homogeneity, the two-point correlation

$$R_{ij}(\mathbf{r}, t) \equiv \langle u_i(\mathbf{x} + \mathbf{r}, t) u_j(\mathbf{x}, t) \rangle, \tag{6.41}$$

is independent of $\mathbf{x}$. At the origin it is

$$R_{ij}(0, t) = \langle u_i u_j \rangle = u'^2 \delta_{ij}. \tag{6.42}$$

There is neither production nor transport, so the evolution equation for the turbulent kinetic energy $k(t) = \frac{3}{2} u'(t)^2$ (Eq. 5.132) reduces to

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\varepsilon. \tag{6.43}$$

As with the structure function D_{ij}, a consequence of isotropy is that R_{ij} can be expressed in terms of two scalar functions $f(r,t)$ and $g(r,t)$:

$$R_{ij}(\mathbf{r}, t) = u'^2 \left\{ g(r,t)\delta_{ij} + [f(r,t) - g(r,t)] \frac{r_i r_j}{r^2} \right\}, \tag{6.44}$$

(cf. Eq. 6.25). With $\mathbf{r} = \mathbf{e}_1 r$, this equation becomes

$$\begin{aligned} R_{11}/u'^2 &= f(r,t) = \langle u_1(\mathbf{x} + \mathbf{e}_1 r, t) u_1(\mathbf{x}, t) \rangle / \langle u_1^2 \rangle, \\ R_{22}/u'^2 &= g(r,t) = \langle u_2(\mathbf{x} + \mathbf{e}_1 r, t) u_2(\mathbf{x}, t) \rangle / \langle u_2^2 \rangle, \\ R_{33} &= R_{22}, \quad \text{and} \quad R_{ij} = 0, \quad \text{for} \quad i \neq j, \end{aligned} \tag{6.45}$$

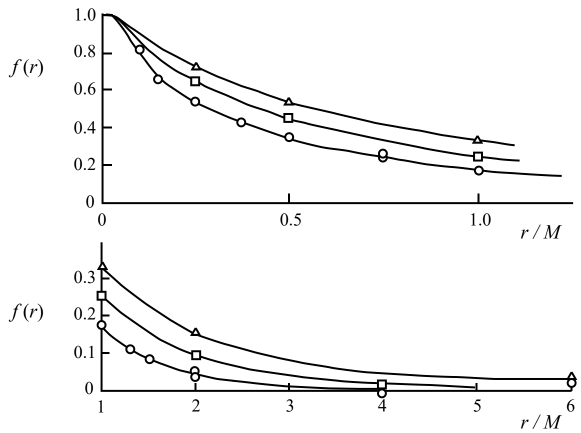

Figure 6.6: Measurements of the longitudinal velocity autocorrelation functions $f(r,t)$ in grid turbulence: $x_1/M = 42, \circ; 98, \square; 172, \triangle$. From Comte-Bellot and Corrsin (1971).

thus identifying f and g as the longitudinal and transverse autocorrelation functions, respectively. (Note that f and g are non-dimensional with $f(0,t) = g(0,t) = 1$.) Again in parallel with the properties of D_{ij}, the continuity equation implies $\partial R_{ij}/\partial r_j = 0$ (see Exercise 3.35) which, in combination with Eq. (6.44), leads to

$$g(r,t) = f(r,t) + \tfrac{1}{2} r \frac{\partial}{\partial r} f(r,t). \tag{6.46}$$

Thus, in isotropic turbulence the two-point correlation $R_{ij}(\mathbf{r},t)$ is completely determined by the longitudinal autocorrelation function $f(r,t)$. (which is only in homogeneous, Figure 6.6 shows the measurements of $f(r,t)$ in nearly isotropic grid-generated turbulence obtained by Comte-Bellot and Corrsin (1971).

There are two distinct longitudinal lengthscales $L_{11}(t)$ and $\lambda_f(t)$ that can be defined from f; and then there are corresponding transverse lengthscales $L_{22}(t)$ and $\lambda_g(t)$ defined from g.

Integral Lengthscales. The first of the lengthscales obtained from $f(r,t)$ is the *longitudinal integral scale*

$$L_{11}(t) \equiv \int_0^\infty f(r,t)\,\mathrm{d}r, \tag{6.47}$$

which we have already encountered (e.g., in Section 5.1, Fig. 5.13 on page 113). The integral scale $L_{11}(t)$ is simply the area under the curve of $f(r,t)$, and so inspection of Fig. 6.6 immediately reveals that L_{11} grows with time (in grid turbulence). As previously observed, L_{11} is characteristic of the larger eddies. In isotropic turbulence, the transverse integral scale

$$L_{22}(t) \equiv \int_0^\infty g(r,t)\,\mathrm{d}r, \tag{6.48}$$

is just half of $L_{11}(t)$ (see Exercise 6.4).

Exercise 6.4 Show that Eq. (6.46) can be rewritten

$$g(r,t) = \tfrac{1}{2}\left\{ f(r,t) + \frac{\partial}{\partial r}[r f(r,t)] \right\}, \tag{6.49}$$

and hence that in isotropic turbulence the transverse integral scale

$$L_{22}(t) \equiv \int_0^\infty g(r,t)\,\mathrm{d}r \tag{6.50}$$

is half of the longitudinal scale, i.e.,

$$L_{22}(t) = \tfrac{1}{2} L_{11}(t). \tag{6.51}$$

Exercise 6.5 Show from Eq. (6.46) that

$$\int_0^\infty r g(r,t)\,\mathrm{d}r = 0, \tag{6.52}$$

(assuming that $f(r,t)$ decays more rapidly than r^{-2} for large r).

Taylor Microscales. The second lengthscale obtained from $f(r,t)$ is the *longitudinal Taylor microscale* $\lambda_f(t)$. Since $f(r,t)$ is an even function of r and no greater than unity, the first derivative at the origin $f'(0,t) = (\partial f/\partial r)_{r=0}$ is zero, while the second derivative $f''(0,t) = (\partial^2 f/\partial r^2)_{r=0}$ is non-positive. As we shall see, in turbulence $f''(0)$ is strictly negative, and so $\lambda_f(t)$ defined by

$$\lambda_f(t) = \left[-\tfrac{1}{2} f''(0,t) \right]^{-\frac{1}{2}}, \tag{6.53}$$

is real, positive and has dimensions of length.

A geometric construction makes this abstruse definition clear. Let $p(r)$ be the osculating parabola to $f(r)$ at $r = 0$ (i.e., the parabola with $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$). Evidently $p(r)$ is

$$\begin{aligned} p(r) &= 1 + \tfrac{1}{2} f''(0) r^2 \\ &= 1 - r^2/\lambda_f^2. \end{aligned} \tag{6.54}$$

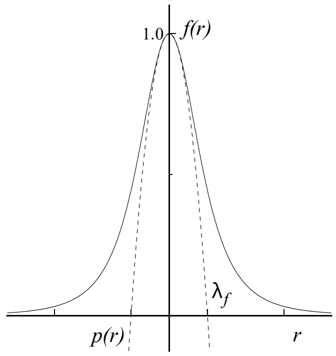

Figure 6.7: Sketch of the longitudinal velocity autocorrelation function showing the definition of the Taylor microscale λ_f.

Thus, as sketched in Fig. 6.7, the osculating parabola intersects the axis at $r = \lambda_f$.

As the following manipulation shows, $f''(0,t)$ (and hence $\lambda_f(t)$) is related to velocity derivatives:

$$\begin{aligned}
-u'^2 f''(0,t) &= -u'^2 \lim_{r\to 0} \frac{\partial^2}{\partial r^2} f(r,t) \\
&= -\lim_{r\to 0} \frac{\partial^2}{\partial r^2} \langle u_1(\mathbf{x} + e_1 r, t) u_1(\mathbf{x}, t)\rangle \\
&= -\lim_{r\to 0} \left\langle \left(\frac{\partial^2 u_1}{\partial x_1^2}\right)_{\mathbf{x}+\mathbf{e}_1 r} u_1(\mathbf{x},t) \right\rangle \\
&= -\left\langle \left(\frac{\partial^2 u_1}{\partial x_1^2}\right) u_1 \right\rangle \\
&= -\left\langle \frac{\partial}{\partial x_1}\left[u_1 \frac{\partial u_1}{\partial x_1}\right] - \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \\
&= \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle .
\end{aligned} \tag{6.55}$$

Thus we obtain

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle = \frac{2u'^2}{\lambda_f^2}. \tag{6.56}$$

The transverse Taylor microscale $\lambda_g(t)$, defined by

$$\lambda_g(t) = \left[-\tfrac{1}{2} g''(0,t)\right]^{-\frac{1}{2}}, \tag{6.57}$$

is, in isotropic turbulence, equal to $\lambda_f(t)/\sqrt{2}$ (see Exercise 6.6). It then follows from these two equations and the relation $\varepsilon = 15\nu\langle(\partial u_1/\partial x_1)^2\rangle$ (Eq. 5.171) that the dissipation is given by

$$\varepsilon = 15\nu u'^2/\lambda_g^2. \tag{6.58}$$

In a classic paper marking the start of the study of isotropic turbulence, Taylor (1935a) defined λ_g and obtained the above equation for ε. He then stated that "λ_g may roughly be regarded as a measure of the diameter of the smallest eddies which are responsible for the dissipation of energy." This deduction from Eq. (6.58) is incorrect, because it incorrectly supposes that u' is the characteristic velocity of the dissipative eddies. Instead, the characteristic length and velocity scales of the smallest eddies are the Kolmogorov scales η and u_η.

To determine the relationship between the Taylor and Kolmogorov scales, we define $L = k^{\frac{3}{2}}/\varepsilon$ to be the lengthscale characterizing the large eddies, and the turbulence Reynolds number to be

$$\mathrm{Re}_L \equiv \frac{k^{\frac{1}{2}}L}{\nu} = \frac{k^2}{\varepsilon\nu}. \tag{6.59}$$

Then the microscales are given by

$$\lambda_g/L = \sqrt{10}\,\mathrm{Re}_L^{-\frac{1}{2}}, \tag{6.60}$$

$$\eta/L = \mathrm{Re}_L^{-\frac{3}{4}}, \tag{6.61}$$

and

$$\lambda_g = \sqrt{10}\,\eta^{\frac{2}{3}}L^{\frac{1}{3}}. \tag{6.62}$$

Thus at high Reynolds number, λ_g is intermediate in size between η and L.

The Taylor scale does not have a clear physical interpretation. It is, however, a well-defined quantity that is often used. In particular, the Taylor-scale Reynolds number

$$\mathrm{R}_\lambda \equiv u'\lambda_g/\nu, \tag{6.63}$$

is traditionally used to characterize grid-turbulence. Observe, from Eq. (6.60), that R_λ varies as the square-root of the integral-scale Reynolds number

$$\mathrm{R}_\lambda = \left(\tfrac{20}{3}\mathrm{Re}_L\right)^{\frac{1}{2}}. \tag{6.64}$$

In addition, it may be observed that the ratio

$$\lambda_g/u' = (15\nu/\varepsilon)^{\frac{1}{2}} = \sqrt{15}\ \tau_\eta, \tag{6.65}$$

correctly characterizes the timescale of the small eddies.

Exercise 6.6 Show from Eq. (6.46) that

$$g''(r,t) = 2f''(r,t) + \tfrac{1}{2} r f'''(r,t), \tag{6.66}$$

and hence that the transverse Taylor microscale

$$\lambda_g(t) \equiv \left[-\tfrac{1}{2} g''(0,t)\right]^{-\frac{1}{2}}, \tag{6.67}$$

is related to the longitudinal scale $\lambda_f(t)$ by

$$\lambda_g(t) = \lambda_f(t)/\sqrt{2}. \tag{6.68}$$

Show

$$\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle = \frac{2u'^2}{\lambda_g^2}. \tag{6.69}$$

Kármán-Howarth Equation. von Kármán and Howarth (1938) obtained from the Navier-Stokes equations an evolution equation for $f(r,t)$. We outline here the principal steps, the result, and some implications: a detailed derivation can be found in the original work or in standard references (e.g., Hinze 1975, Monin and Yaglom 1975).

The time derivative of $R_{ij}(\mathbf{r},\mathbf{x},t)$ can be expressed as

$$\begin{aligned}
\frac{\partial}{\partial t} R_{ij}(\mathbf{r},t) &= \frac{\partial}{\partial t}\langle u_i(\mathbf{x}+\mathbf{r},t) u_j(\mathbf{x},t)\rangle \\
&= \left\langle u_j(\mathbf{x},t)\frac{\partial}{\partial t} u_i(\mathbf{x}+\mathbf{r},t)\right\rangle \\
&+ \left\langle u_i(\mathbf{x}+\mathbf{r},t)\frac{\partial}{\partial t} u_j(\mathbf{x},t)\right\rangle,
\end{aligned} \tag{6.70}$$

and then the Navier-Stokes equations, i.e.,

$$\frac{\partial u_j}{\partial t} = -\frac{\partial (u_i u_j)}{\partial x_i} - \frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i}, \tag{6.71}$$

can be used to eliminate the time derivatives on the right-hand side of Eq. (6.70). Three types of terms arise, corresponding to the convection, pressure-gradient, and viscous terms in Eq. (6.71). For isotropic turbulence the pressure-gradient term in the equation for $R_{ij}(\mathbf{r},t)$ is zero.

The convective term involves two-point triple velocity correlations, such as

$$\bar{S}_{ijk}(\mathbf{r},t) \equiv \langle u_i(\mathbf{x},t)u_j(\mathbf{x},t)u_k(\mathbf{x}+\mathbf{r},t)\rangle. \tag{6.72}$$

Just as R_{ij} is uniquely determined by f (Eq. 6.44), in isotropic turbulence $\bar{S}_{ijk}$ is uniquely determined by the longitudinal correlation

$$\begin{aligned} \bar{k}(r,t) &= \bar{S}_{111}(\mathbf{e}_1 r,t)/u'^3 \\ &= \langle u_1(\mathbf{x},t)^2 u_1(\mathbf{x}+\mathbf{e}_1 r,t)\rangle/u'^3. \end{aligned} \tag{6.73}$$

It can be shown that $\bar{k}(r,t)$ is an odd function of r, and that the continuity equation implies $\bar{k}'(0,t)=0$, so that its series expansion is

$$\bar{k}(r,t) = \bar{k}'''r^3/3! + \bar{k}^{\mathrm{v}}r^5/5! \ldots. \tag{6.74}$$

By this procedure, an exact equation for $f(r,t)$ is obtained from the Navier-Stokes equations: it is the *Kármán-Howarth equation*

$$\frac{\partial}{\partial t}(u'^2 f) - \frac{u'^3}{r^4}\frac{\partial}{\partial r}(r^4\bar{k}) = \frac{2\nu u'^2}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right). \tag{6.75}$$

The principal observations to be made are:

(i) There is a closure problem. This single equation involves two unknown functions $f(r,t)$ and $\bar{k}(r,t)$.

(ii) The terms in $\bar{k}$ and ν represent inertial and viscous processes, respectively.

(iii) At $r=0$, the term in $\bar{k}$ vanishes (on account of Eq. 6.74); while, from the fact that f is even in r, we obtain

$$\left[\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)\right]_{r=0} = 5f''(0,t) = -\frac{5}{\lambda_g(t)^2}. \tag{6.76}$$

Hence, for $r=0$, the Kármán-Howarth equation reduces to ($\frac{2}{3}$ times) the kinetic energy equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}u'(t)^2 = -10\nu\frac{u'^2}{\lambda_g^2} = -\tfrac{2}{3}\varepsilon. \tag{6.77}$$